

Regge Calculus in Teleparallel Gravity

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Abstract.

In the context of the teleparallel equivalent of general relativity, the Weitzenböck manifold is considered as the limit of a suitable sequence of discrete lattices composed of an increasing number of smaller and smaller simplices, where the interior of each simplex (Delaunay lattice) is assumed to be flat. The link lengths l between any pair of vertices serve as independent variables, so that torsion turns out to be localized in the two dimensional hypersurfaces (dislocation triangle, or hinge) of the lattice. Assuming that a vector undergoes a dislocation in relation to its initial position as it is parallel transported along the perimeter of the dual lattice (Voronoi polygon), we obtain the discrete analogue of the teleparallel action, as well as the corresponding simplicial vacuum field equations.

PACS numbers: 04.20.-q; 04.50.+h; 04.60.Nc

1. Introduction

Regge calculus is a useful tool for numerical calculation in curved spacetimes, and has been applied to a variety of problems in cosmology, as well as in classical and quantum gravity [1]. In the well known paper by Regge [2], the usual continuous space-time manifold of general relativity is viewed as the limit of a suitable sequence of discrete lattices composed of an increasing number of smaller and smaller simplices. In this approach, the interior of each simplex is assumed to be flat, and the link lengths l between any pair of its vertices serve as independent variables so that the scalar curvature is defined as a distribution with support on the bones of the lattice.

Few attempts have been made to include concepts of non-riemannian geometry into the Regge calculus [3, 4, 5, 6, 7, 8]. In some of these attempts [4, 7], the Regge calculus was formulated as a lattice gauge theory for the Poincaré group, and the possibility of including torsion as closure failures of the building blocks of the simplicial manifold was pointed out. Another of such attempt was an application to the Einstein-Cartan theory [8], where the square of the torsion tensor was defined as a distribution, and the corresponding simplicial field equations were obtained. The notion of torsion singularities appearing as a conical defect (dislocation) has also been discussed in the literature [9]. Its application to the theory of solid state and crystal defects has been known for a long time [10, 11, 12, 13, 14], and the connection to the low dimensional gravity has already been pointed out [15, 16, 17].

An alternative approach to gravitation is the so called teleparallel gravity [18], which is gauge theory based on the Weitzenböck geometry [19]. In this theory, gravitation is attributed to torsion [20], which plays the role of a force [21], and the curvature tensor vanishes identically. As is well known, at least in the absence of spinor fields, teleparallel gravity is equivalent to general relativity. In this paper, relying upon this equivalence, we will construct the discrete counterpart of the four-dimensional teleparallel action, and will obtain the corresponding field equations in vacuum. In a vacuum simplicial Weitzenböck geometry, the torsion tensor is localized in the two-dimensional dislocation triangles, called hinges. Torsion can then be detected by measuring the dislocation in relation to the initial position of a vector, as a result of the parallel transport along a small loop encircling the dislocation triangle (hinge), where torsion is concentrated. When torsion is present, it is detected parallel to this hinge, and the dislocation is measured by the Burgers vector b_d . The metric properties of this simplicial manifold are given by the lengths l_p of the edges. This means that the variation of the simplicial action is to be made with respect to l_p . The resulting simplicial field equations can be considered as the teleparallel equivalent of the simplicial Einstein's equation of general relativity, first obtained by Regge [2].

It should be remarked that the Regge calculus has already been applied to the Einstein-Cartan theory [8], where the fermions act as a source of torsion. In this case, the Burgers vector (b_d) couples algebraically to the matter term, and so the variation of the action is to be made with respect to both the edge lengths and the Burgers vectors.

Furthermore, as in the absence of matter no dislocations can be present on the lattice, the Burgers vector will be zero, and the simplicial field equations will reduce to the usual Regge's equations of general relativity. As we are going to see, in the teleparallel approach, even in the presence of matter, the Burgers vector does not couple to matter. Furthermore, in the absence of matter, the Burgers vector does not vanish, a property which is in accordance with the fact that, in teleparallel gravity, torsion is a propagating field. We will proceed according to the following scheme. In section 2, we review the main features of teleparallel gravity. In section 3, we obtain the simplicial torsion, the discrete action, and the simplicial field equation. Discussions and conclusions are presented in section 4.

2. Teleparallel Equivalent of General Relativity

It is well known that curvature, according to general relativity, is used to geometrize the gravitational interaction. On the other hand, teleparallelism attributes gravitation to torsion, but in this case torsion accounts for gravitation not by geometrizing the interaction, but by acting as a force [21]. This means that in the teleparallel equivalent of general relativity, instead of geodesics, there are force equations quite analogous to the Lorentz force equation of electrodynamics.

A nontrivial tetrad field induces on spacetime a teleparallel structure which is directly related to the presence of the gravitational field. In this case, tensor and local Lorentz indices[‡] can be changed into each other with the use of a tetrad field h^a_μ . A nontrivial tetrad field can be used to define the linear Weitzenböck connection

$$\Gamma^\sigma_{\mu\nu} = h_a^\sigma \partial_\nu h^a_\mu, \quad (1)$$

a connection presenting torsion, but no curvature. It parallel transports the tetrad itself:

$$\nabla_\nu h^a_\mu \equiv \partial_\nu h^a_\mu - \Gamma^\rho_{\mu\nu} h^a_\rho = 0. \quad (2)$$

The Weitzenböck connection satisfies the relation

$$\Gamma^\sigma_{\mu\nu} = \overset{\circ}{\Gamma}^\sigma_{\mu\nu} + K^\sigma_{\mu\nu}, \quad (3)$$

where

$$\overset{\circ}{\Gamma}^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\rho} [\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}] \quad (4)$$

is the Levi-Civita connection of the metric

$$g_{\mu\nu} = \eta_{ab} h^a_\mu h^b_\nu, \quad (5)$$

and

$$K^\sigma_{\mu\nu} = \frac{1}{2} [T^\sigma_{\mu\nu} + T^\sigma_{\nu\mu} - T^\sigma_{\mu\mu}] \quad (6)$$

[‡] The greek alphabet ($\mu, \nu, \rho, \dots = 0, 1, 2, 3$) will be used to denote tensor indices, that is, indices related to spacetime. The latin alphabet ($a, b, c, \dots = 0, 1, 2, 3$) will be used to denote local Lorentz (or tangent space) indices, whose metric tensor is chosen to be $\eta_{ab} = \text{diag}(+1, -1, -1, -1)$. Furthermore, we will use units in which $\hbar = c = 1$.

is the contorsion tensor, with

$$T^\sigma{}_{\mu\nu} = \Gamma^\sigma{}_{\nu\mu} - \Gamma^\sigma{}_{\mu\nu} \quad (7)$$

the torsion of the Weitzenböck connection [22].

The teleparallel gravitational lagrangian is

$$L_G = \frac{h}{16\pi G} \left[\frac{1}{4} T^\rho{}_{\mu\nu} T^\mu{}_{\rho}{}^{\nu} + \frac{1}{2} T^\rho{}_{\mu\nu} T^{\nu\mu}{}_{\rho} - T_{\rho\mu}{}^{\rho} T^{\nu\mu}{}_{\nu} \right], \quad (8)$$

where $h = \det(h^a{}_\mu)$. By considering now the functional variation of L_G in relation to $h^a{}_\mu$, we obtain the teleparallel gravitational field equation [23]

$$\partial_\sigma (h S_\lambda{}^{\sigma\tau}) - 4\pi G (h t_\lambda{}^\tau) = 0, \quad (9)$$

where

$$t_\lambda{}^\tau = \frac{1}{4\pi G} \Gamma^\mu{}_{\nu\sigma} S_\mu{}^{\sigma\tau} + \delta_\lambda{}^\tau L_G, \quad (10)$$

is the canonical energy-momentum pseudo tensor of the gravitational field, and $S^{\rho\mu\nu}$ is the tensor

$$S^{\rho\mu\nu} = \frac{1}{2} \left[K^{\mu\nu\rho} - g^{\rho\nu} T^{\theta\mu}{}_\theta + g^{\rho\mu} T^{\theta\nu}{}_\theta \right]. \quad (11)$$

3. Discrete torsion and simplicial field equation

We are going to obtain now the simplicial teleparallel action, as well as the corresponding field equation. In a heuristic way, and similarly to the Regge construction of the simplicial manifold of general relativity, we assume that the usual continuous spacetime manifold of teleparallel gravity can be viewed as the limit of a suitable sequence of discrete lattices composed of an increasing number of smaller and smaller simplices. In other words, the Weitzenböck manifold, which is the stage set of teleparallel gravity, is approximated by a four-dimensional polyhedra M^4 . In this approach, the interior of each simplex is assumed to be flat, and this flat four-simplices are joined together at the tetrahedral faces of their boundaries. The torsion turns out to be localized in the two-dimensional dislocation triangles (hinges) of the lattice, and the link lengths l between any pair of vertices serve as independent variables. By varying the action with respect to this edge lengths, one obtains the simplicial analogues of the teleparallel field equations.

Let us then proceed to this construction. To begin with, let us take a bundle of parallel dislocations (hinges) in M^3 . We make the assumption that the torsion induced by the dislocations is small, so that we may regard M^3 as approximately euclidian. Let \mathbf{U} be a unity vector parallel to the dislocations. We test for the presence of torsion by carrying a vector \mathbf{A} around a small loop of area vector $\mathbf{S} = S\mathbf{n}$, with S the area and \mathbf{n} a unity vector normal to the surface. At the end of the test, if torsion is nonvanishing, \mathbf{A} is found to have translated from the original position, along \mathbf{U} , by the length $\mathbf{B} = N\mathbf{b}$, where N is the number of dislocations entangled by the loop, and \mathbf{b} is the Burgers vector, which is a vector that gives both the length and direction of the closure failure

for every dislocation. In M^4 , the flux of dislocation lines through the loop of area $S^{\alpha\beta}$ is

$$\Phi = \rho(\mathbf{US}) = \frac{1}{2} \rho_{\alpha\beta} S^{\alpha\beta},$$

where ρ is the density of dislocation passing through the loop, and $\rho_{\alpha\beta} = \rho U_{\alpha\beta}$, with $U^{\alpha\beta}$ a unity antisymmetric tensor satisfying $U_{\alpha\beta} U^{\alpha\beta} = 2$. This means that we can endow the polyhedra more densely with hinges in a region of high torsion than in region of low torsion. The closure failure is then found to be

$$B_\mu = \frac{1}{2} \rho_{\alpha\beta} S^{\alpha\beta} b_\mu. \quad (12)$$

However, we know from differential geometry that, in the presence of torsion, infinitesimal parallelograms in spacetime do not close, the closure failure being equal to

$$B_\mu = T_{\mu\nu\sigma} S^{\nu\sigma}. \quad (13)$$

By comparing the last two equations we see that

$$T_{\mu\alpha\beta} = \frac{1}{2} \rho_{\alpha\beta} b_\mu \equiv \frac{1}{2} \rho U_{\alpha\beta} b_\mu. \quad (14)$$

On the other hand, it has already been shown [9] that the torsion singularity takes the form of a conical singularity. Consequently, the dislocation from the original position that occurs when a vector is parallel transported around a small loop encircling a given bone is independent of the area of the loop. Furthermore, the dislocation has two characteristics: The length of the dislocation line in three-dimensions, and the area of the triangle in four-dimensions. Therefore, there is a natural unique volume associated with each dislocation. To define this volume, there is a well-known procedure in which a dual lattice is constructed for any given lattice [24, 25]. This involves constructing polyhedral cells around each vertex, known in the literature as Voronoi polygon, in such a way that the polygon around each particular vertex contains all points which are nearer to that vertex than to any other vertex. The boundary of the Voronoi polygon is always perpendicular to the edges emanating from the vertex, and each corner of the Voronoi polygon lies at the circumcentre of any of the simplices of the Delaunay geometry, which shares the dislocation (bone) (see Fig. 1).

By construction, the Voronoi polygon is orthogonal to the bone. If we parallel transport a vector around the perimeter of a Voronoi polygon of area Σ_d^* , it will traverse the flat geometry of the interior of each one of the simplices sharing the bone, and will return dislocated from its original position in a plane parallel to the bone by a length b_μ . According to this construction, and relying on the definition (14), the torsion due to each dislocation can be expressed by (see Fig. 2):

$$(\text{Torsion}) = \frac{(\text{Distance the vector is translated})}{(\text{Area circumnavigated})}. \quad (15)$$

This definition is equivalent to the definition of Gauss curvature:

$$(\text{Gauss curvature}) = \frac{(\text{Angle the vector is rotated})}{(\text{Area circumnavigated})}. \quad (16)$$

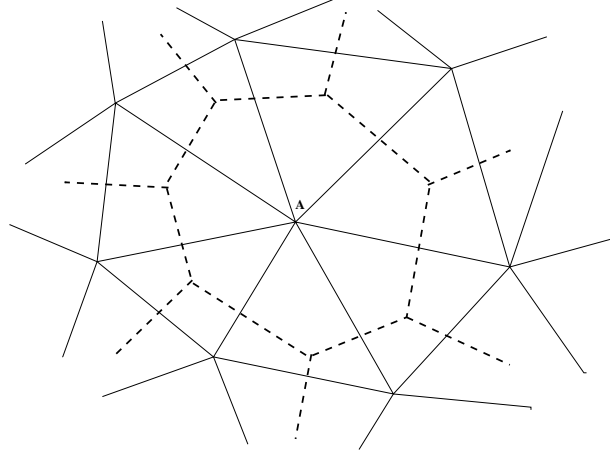


Figure 1. The two-dimensional Voronoi polygon (dashed line) for a particular vertex A . Each corner of the Voronoi polygon lies at the circumcentre of any of the triangles of the Delaunay geometry (solid line).

Therefore, analogously to the Riemann scalar, which is proportional to the Gauss curvature, and whose proportionality constant depends on the dimension D of the lattice geometry [25], we define the simplicial torsion due to each dislocation by

$$T_{(d)\mu\nu\rho} = \sqrt{D(D-1)} \frac{b_{(d)\mu} U_{(d)\nu\rho}}{\Sigma_d^*} \equiv \sqrt{12} \frac{b_{(d)\mu} U_{(d)\nu\rho}}{\Sigma_d^*}. \quad (17)$$

The reason for the square root is that, as is well known from teleparallel gravity, the Riemann curvature tensor is proportional to a combination of squared torsion tensors.

As we have said, the vector returns translated from its original position in a plane parallel to the hinge by a length b^μ . Let us then analyze the translational group acting on it. As we know, the interior of each block is flat (Minkowski space), so the infinitesimal translation in these blocks is given by

$$T(\delta b) = I - i \delta b^a P_a, \quad (18)$$

where I is the unity matrix, δb^a are the components of an arbitrarily small four-dimensional Burgers (displacement) vector, and $P_a = i\partial_a$ are the translation generators. In the presence of dislocations, and using the tetrad h^a_μ , this infinitesimal translation parallel to the hinge becomes

$$T(\delta b) = I - i \delta b^\mu h^a_\mu P_a, \quad (19)$$

so that a finite translation will be represented by the group element

$$T(b) = \exp[-i b^\mu h^a_\mu P_a]. \quad (20)$$

On the other hand, the contour integral of the Burgers vector — which measures how much the infinitesimal closed contour Γ spanning a surface element $d\Sigma^{*\alpha\beta}$ fails to close in the presence of hinge — by using Eq. (13), is seen to be [26]

$$b^\mu = \oint_{\Gamma} T^\mu_{\alpha\beta} d\Sigma^{*\alpha\beta}. \quad (21)$$

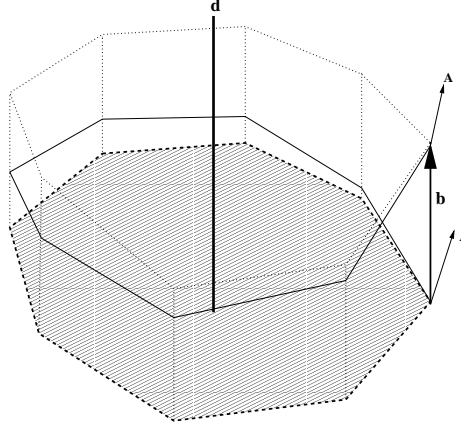


Figure 2. The parallel transport of a vector A along the perimeter of a Voronoi polygon (dashed line) around the dislocation line in three-dimension. The vector returns translated in a plane parallel to the dislocation d by the length b .

Therefore, the group element of translations due to torsion turns out to be

$$T(b) = \exp \left[-i \oint_{\Gamma} P_a T^a_{\alpha\beta} d\Sigma^{*\alpha\beta} \right]. \quad (22)$$

The four-volume Ω_d associated with each dislocation, as described above, is proportional to the product of Σ_d , the two-dimensional volume of the dislocation, and Σ_d^* , the area of the Voronoi polygon [25]:

$$\Omega_d \equiv \frac{2}{D(D-1)} \Sigma_d \Sigma_d^* = \frac{1}{6} \Sigma_d \Sigma_d^*. \quad (23)$$

The invariant volume element $h d^4x$, therefore, is represented by Ω_d , and we have the following relation,

$$\int h d^4x \implies \sum_{\text{dis}} \Omega_d = \frac{1}{6} \sum_{\text{dis}} \Sigma_d \Sigma_d^*, \quad (24)$$

where the sum is made over all dislocations. We are ready then for constructing the simplicial action. Let us take the lagrangian (8) of teleparallel gravity, whose terms are proportional to the square of the torsion tensor, and substitute torsion as given by Eq. (17). For the first term, we obtain

$$T_{(d)}^{\mu\nu\rho} T_{(d)\mu\nu\rho} = 24 \left(\frac{1}{\Sigma_d^*} \right)^2 b_{(d)}^\mu b_{(d)\mu}. \quad (25)$$

Writing the other two terms in a similar way, the simplicial teleparallel action will be

$$S = \frac{1}{16\pi G} \sum_{\text{dis}} \left(\frac{b_d^2}{\Sigma_d^*} \right) \Sigma_d, \quad (26)$$

where b_d^2 denotes the projected Burgers vector parallel to the hinge, and Σ_d is the area of this dislocation triangle, or hinge.

On the other hand, the simplicial Einstein-Hilbert action of general relativity is [2]

$$S = \frac{1}{8\pi G} \sum_{\text{hinge}} \varepsilon_h \Sigma_h, \quad (27)$$

where ε_h is the deficit angle associated to each hinge, which is directly related to the curvature of spacetime. Now, up to a surface term, the Einstein-Hilbert action of general relativity is known to be equivalent to the teleparallel action (8). Assuming that the corresponding simplicial versions of the same actions are also equivalent, we can obtain a relation between the *angle* ε_h , through which a vector is *rotated* in relation to its initial direction when parallel transported around a small loop, and the *distance* b_d through which a vector is *translated* from its original position when parallel transported around a small loop. By comparing Eqs. (26) and (27), this relation is found to be

$$\varepsilon_h = \frac{1}{2} \left(\frac{b_d^2}{\Sigma_d^*} \right). \quad (28)$$

Let us now carry out the variation of the simplicial action (26) with respect to the edge lengths l_p . In the usual application of Regge calculus to a curved spacetime, the typical deficit angle ε_h related to each hinge depends in a complicated trigonometric way on the values of numerous edge lengths l_p . However, as demonstrated by Regge [2], one can carry out the variation with respect to the edge lengths l_p as if the ε_h were constants. Similarly, in the teleparallel simplicial gravity, both the projected Burgers vector b_d^2 and the area of the dual Voronoi polygon Σ_d^* related to each hinge depend also in a complicated trigonometric way on the values of numerous edge lengths l_p . Because of the relation (28), however, in the same way ε_h can be considered as a constant in the variation of the simplicial Einstein-Hilbert action, one can equivalently carry out the variation of the teleparallel simplicial action (26) as if the dimensionless parameters (b_d^2/Σ_d^*) were constants. This is the teleparallel version of the result demonstrated by Regge. We thus find

$$\delta S = \frac{1}{16\pi G} \sum_{\text{dis}} \left(\frac{b_d^2}{\Sigma_d^*} \right) \delta \Sigma_d. \quad (29)$$

The change in the area of the triangle hinge, on the other hand, is given by

$$\delta \Sigma_d = \frac{1}{2} \sum_p l_p \delta l_p \cot \theta_{(dp)}, \quad (30)$$

where $\theta_{(dp)}$ is the angle in the triangle hinge d opposite to the edge l_p . The simplicial teleparallel field equation in vacuum then reads

$$l_p \sum_{d \supset l_p} \left(\frac{b_d^2}{\Sigma_d^*} \right) \cot \theta_{(dp)} = 0, \quad (31)$$

where the sum is made over all dislocation triangles d which have the given edge p in common.

The simplicial equation (31) is a set of equations for the edge lengths l_p , and consequently there will be one simplicial equation for each edge in the lattice. To solve it, initial conditions consisting of informations about some of the edge lengths

must be supplied. Then, from inputs expressing conditions on the torsion, that is, on the Burgers vector associated to each hinge, one can determine the remainder of the edge lengths, and consequently the geometrical properties of the simplicial Weitzenböck manifold. This is quite similar to simplicial general relativity, where the inputs are conditions on the curvature, that is, on the deficit angle associated to each hinge. It should be remarked, however, that despite being as numerous as the edges, the simplicial equations are not able in general to determine all edge lengths. The problem is that there are relations between these equations that prevent them from being independent, and in consequence, they cannot determine all the details of the largely arbitrary lattice. Therefore, to prepare a real computer program based on Regge calculus, one has to supply the computer not only with the simplicial equation and initial conditions, but also with definite algorithms to remove all the arbitrariness related to the lattice being used [27].

4. Final Remarks

We have considered in this paper the Weitzenböck geometry as the limit of a suitable sequence of discrete lattices, the interior of each simplex being assumed to be flat, and the torsion localized in the two-dimensional dislocation triangles (hinges) of the lattice. The information about the metric properties is codified in the link lengths l_p between any pair of vertices. Furthermore, from the fact that a parallel transported vector undergoes a translation from its original position as it is transported along the perimeter of the Voronoi polygon, we have found the simplicial torsion. Analyzing the infinitesimal translations in flat lattices, the translational group due to torsion was also found. Then, by using the simplicial torsion, we have constructed the simplicial version of the teleparallel action, and finally, by varying the simplicial action with respect to the edge lengths, we have found the simplicial field equations in vacuum.

In the simplicial Einstein-Cartan approach, the fermions act as a source of torsion, and the Burgers vector couples to the matter term. In the absence of matter, there is no dislocation on the lattice, and the Burgers vector is consequently zero. In this case, the simplicial field equations reduce to the Regge's equation of general relativity. On the other hand, in our approach, the geometrical properties of the vacuum simplicial Weitzenböck manifold is replaced by the lengths l_p of the edges, which means that the variation of the simplicial action is to be made with respect to l_p as if both the Burgers vector and the area of the Voronoi dual polygon were constants. The corresponding simplicial field equations turns out to be the teleparallel equivalent of the Regge's general relativity equations. It should be remarked that in this approach, even in the presence of matter, the Burgers vector does not couple to matter, and the variation is made only with respect to l_p . Furthermore, differently from the simplicial Einstein-Cartan approach, in the absence of matter, the Burgers vector does not need to vanish, a property that reflects the fact that, in teleparallel gravity, torsion is a propagating field. Due to its gauge structure, teleparallel gravity turns out to be quite similar to the

other known gauge theories, and depending on the problem to be considered, it can present several formal advantages in relation to general relativity. In this connection, we hope that these results can be useful for numerical calculations in different contexts like gravitation, cosmology, as well as quantum gravity.

Acknowledgments

The authors would like to thank Yu. N. Obukhov for useful comments. They would like to thank also FAPESP-Brazil, CNPq-Brazil and CAPES-Brazil for financial support.

References

- [1] Williams R M and Tuckey P A 1992 *Class. Quantum Grav.* **9**, 1409
- [2] Regge T 1961 *Nuovo Cim.* **19** 558
- [3] Drummond I T 1986 *Nucl. Phys. B* **273**, 125
- [4] Caselle M, Dadda A and Magnea L 1989 *Phys. Lett. B* **232**, 457
- [5] Holm C and Hennig J D 1991 in *Classical and Quantum Systems - Foundations and Symmetries* Proceedings of the II International Wigner Symposium, Goslar, 1991, ed H D Doebner, W Scherer, F Schroeck Jr. (Singapore: World Scietific)
- [6] Holm C and Hennig J D 1991 in *Group Theoretical Methods in Physics* Lecture Notes in Physics 382, ed V V Dodonov and V I Manko (Berlin: Springer)
- [7] Gronwald F 1995 *Class. Quantum Grav.* **12**, 1181
- [8] Schmidt J and Kohler C 2001 *Gen. Rel. Grav.* **33**, 1799
- [9] Tod K P 1994 *Class. Quantum Grav.* **11**, 1331
- [10] Kondo K 1952 *On the Geometrical and Physical Foundation of the Theory of Yielding*, in Proceedings of the 2nd Japan National Congress for Applied Mechanics, page 41
- [11] Bilby B, Bullough R and Smith E 1960 *Proc. Roy. Soc. A* **231**, 263
- [12] Kroner E 1960 *Arch. Ration. Mech. Anal.* **4**, 467
- [13] Gairola B D K 1965 *Nonlinear Elastic Problems*, in Dislocation in Solids, Vol. 1, ed. F R N Nabarro (Amsterdam: North Holland)
- [14] Holz A 1988 *Class. Quantum Grav.* **5**, 1259
- [15] Holz A 1992 *J. Phys. A* **25**, L1
- [16] Katanaev M O and Volovich I V 1992 *Ann. Phys.* **216**, 1
- [17] Kohler C 1995 *Class. Quantum Grav.* **12**, L11
- [18] The basic references on teleparallel gravity can be found in Gronwald F and Hehl F W 1996 in Proc. School of Cosmology and Gravitation on Quantum Gravity, Erice, 1995, ed P G Bergmann, V de Sabbata and H J Treder (Singapore: World Scientific); see also Mueller-Hoissen F and Nitsch J 1983 *Phys. Rev. D* **28**, 718, and references therein
- [19] Weitzenböck R 1923 *Invariantentheorie* (Gronningen: Noordhoff)
- [20] Hayashi K and Shirafuji T 1976 *Phys. Rev. D* **19**, 3524
- [21] de Andrade V C and Pereira J G 1997 *Phys. Rev. D* **56**, 4689
- [22] Aldrovandi R and Pereira J G 1995 *An Introduction to Geometrical Physics* (Singapore: World Scientific)
- [23] de Andrade V C, Guillen L C T and Pereira J G 2000 *Phys. Rev. Lett.* **84**, 4533
- [24] Christ N, Friedberg R and Lee T D 1982 *Nucl. Phys. B* **202**, 89; Hamber H W and Williams R M 1984 *Nucl. Phys. B* **248**, 392
- [25] Miller W A 1997 *Class. Quantum Grav.* **14**, L199
- [26] Kleinert H 1989 *Gauge Fields in Condensed Matter*, Volume II (Singapore: World Scientific)
- [27] See, for example, Misner C W, Thorne K S and Wheeler J A 1973 *Gravitation* (New York: Freeman)